

# Structure of Singularities of 3D Axi-symmetric Navier-Stokes Equations

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## Abstract

Let  $v$  be a solution of the axially symmetric Navier-Stokes equation. We determine the structure of certain (possible) maximal singularity of  $v$  in the following sense. Let  $(x_0, t_0)$  be a point where the flow speed  $Q_0 = |v(x_0, t_0)|$  is comparable with the maximum flow speed at and before time  $t_0$ . We show after a space-time scaling with the factor  $Q_0$  and the center  $(x_0, t_0)$ , the solution is arbitrarily close in  $C_{\text{local}}^{2,1,\alpha}$  norm to a nonzero constant vector in a fixed parabolic cube, provided that  $r_0 Q_0$  is sufficiently large. Here  $r_0$  is the distance from  $x_0$  to the  $z$  axis. Similar results are also shown to be valid if  $|r_0 v(x_0, t_0)|$  is comparable with the maximum of  $|rv(x, t)|$  at and before time  $t_0$ .

## 1 Introduction

In this paper we study the structure, in a space time region with maximum flow speed, of solutions to the three dimensional incompressible Navier-Stokes equations

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = \mu \Delta v, \\ \nabla \cdot v = 0, \end{cases} \quad t \geq 0, \quad x \in \mathbb{R}^3 \quad (1.1)$$

with the axially symmetric initial data

$$a(x) = a^r(r, z, t)e_r + a^\theta(r, z, t)e_\theta + a^z(r, z, t)e_z. \quad (1.2)$$

In cylindrical coordinates, the solution  $v = v(x, t)$  is of the form

$$v(x, t) = v^r(r, z, t)e_r + v^\theta(r, z, t)e_\theta + v^z(r, z, t)e_z. \quad (1.3)$$

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Here  $x = (x_1, x_2, z)$ ,  $r = \sqrt{x_1^2 + x_2^2}$  and

$$e_r = \begin{pmatrix} \frac{x_1}{r} \\ \frac{x_2}{r} \\ 0 \end{pmatrix}, \quad e_\theta = \begin{pmatrix} -\frac{x_2}{r} \\ \frac{x_1}{r} \\ 0 \end{pmatrix}, \quad e_z = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \quad (1.4)$$

are the three orthogonal unit vectors along the radial, the angular, and the axial directions respectively. Moreover, the angular, swirl and axial components  $v^r$ ,  $v^\theta$  and  $v^z$  of the velocity field are solutions of ASNS i.e. the axially symmetric Navier-Stokes equations

$$\begin{cases} \partial_t v^r + b \cdot \nabla v^r - \frac{(v^\theta)^2}{r} + \partial_r p = \left(\Delta - \frac{1}{r^2}\right) v^r, \\ \partial_t v^\theta + b \cdot \nabla v^\theta + \frac{v^r v^\theta}{r} = \left(\Delta - \frac{1}{r^2}\right) v^\theta, \\ \partial_t v^z + b \cdot \nabla v^z + \partial_z p = \Delta v^z, \\ b = v^r e_r + v^z e_z, \quad \nabla \cdot b = \partial_r v^r + \frac{v^r}{r} + \partial_z v^z = 0. \end{cases} \quad (1.5)$$

Here without loss of generality, we set the viscosity constant  $\mu = 1$ .

The axially symmetric case appears much special than the full Navier-Stokes equations, however the main regularity problem is just as wide open. Let us briefly discuss some recent interesting results on the axially symmetric Navier-Stokes equations. When  $v^\theta = 0$ , i.e. in the no swirl case, O. A. Ladyzhenskaya [9], and M. R. Uchoviskii & B. I. Yudovich [13] proved that weak solutions are regular for all time. See also the work by S. Leonardi, J. Malek, J. Necas, & M. Pokorný [10]. More recent activities, in the presence of swirl, include the results of C.-C. Chen, R. M. Strain, T.-P. Tsai, & H.-T. Yau in [2] & [3], where they prove that suitable axially symmetric solutions bounded by  $C r^{-\alpha} \sqrt{|t|}^{-1+\alpha}$  ( $0 \leq \alpha \leq 1$ ) are smooth. Here  $r$  is the distance from a point to the  $z$  axis, and  $t$  is time. See also the work of G. Koch, N. Nadirashvili, G. Seregin, & V. Sverak [8] and its local version in G. Seregin & V. Sverak [11] by different methods. Also in the presence of swirl, there is the paper by J. Neustupa & M. Pokorný [6], proving the regularity of one component (either  $v^r$  or  $v^\theta$ ) implies regularity of the other components of the solution. Also proving regularity is the work of Q. Jiu & Z. Xin [7] under an assumption of sufficiently small zero dimension scaled norms. We would also like to mention the regularity results of D. Chae & J. Lee [1] who prove regularity results assuming finiteness of another zero dimensional integral. On the other hand, G. Tian & Z. Xin [12] constructed a family of singular axis symmetric solutions with singular initial data; T. Hou & C. Li [4] found a special class of global smooth solutions. See also a recent extension: T. Hou, Z. Lei & C. Li [5].

In this paper, we take another approach to ASNS, which aims at the understanding of the local structure of solutions when the flow velocity is very high. This approach is more akin to the one taken by Hamilton and Perelman in the study of Ricci flow. We are able to do so when the flow speed  $|v(x_0, t_0)|$  at a space time point  $(x_0, t_0)$  is comparable with the maximum flow speed, or  $r_0 |v(x_0, t_0)|$  at a space time point  $(x_0, t_0)$  is comparable with the maximum of  $r |v(x, t)|$ , at and before time  $t_0$ .

In order to present the result, we introduce some notations. Let  $v = v(x, t)$  be a solution to ASNS which is used here and later to denote axially symmetric Navier-Stokes equations. Here  $(x, t)$  is a point in space time. Given a number  $a > 0$ , and  $(x_0, t_0)$  be a point in space time, we use the following symbol to denote the parabolic cube

$$P(x_0, t_0, a) \equiv \{(x, t) \mid |x_0 - x| < a, t_0 - a^2 \leq t \leq t_0\}.$$

Unless stated otherwise, we use  $r, r_0, r_k$  to denote the distance between points  $x, x_0, x_k$  in space and the  $z$  axis respectively.

Now we are ready to state the main result of the paper.

**Theorem 1.1.** *Let  $v = v(x, t)$ ,  $(x, t) \in \mathbb{R}^3 \times [0, T_0)$ ,  $T_0 > 0$  be a smooth solution to the three-dimensional ASNS whose initial condition  $v_0$  satisfies*

$$\|v_0\|_{L^\infty(\mathbb{R}^3)} \leq N_0, \quad \|v_0\|_{L^2(\mathbb{R}^3)} \leq N_0, \quad |rv_0| \leq N_0. \quad (1.6)$$

Here  $N_0$  is any given positive number. For any sufficiently small constant  $\epsilon > 0$  and another constant  $\sigma_0$ , there exists some  $\rho_0 = \rho_0(\epsilon, N_0, \sigma_0) > 0$  with the following properties.

(a). Suppose

$$r_0|v(x_0, t_0)| \geq \rho_0^{-2}$$

at some point  $(x_0, t_0)$  where  $x_0 \in \mathbb{R}^3$  and  $t_0 \in (0, T_0)$ . Suppose also  $(x_0, t_0)$  is an almost maximal point in the sense:

$$|v(x_0, t_0)| \geq \frac{1}{4} \sup_{x \in \mathbb{R}^3, t \leq t_0} |v(x, t)|.$$

Then the velocity  $v$  in the cube

$$P(x_0, t_0, (\sigma_0 \epsilon Q)^{-1}), \quad Q \equiv |v(x_0, t_0)|,$$

is, after scaling by the factor  $Q$ ,  $\epsilon$  close in  $C_{\text{local}}^{2,1,\alpha}$  norm to a nonzero constant vector.

(b). The conclusion in (a) still holds if

$$r_0|v(x_0, t_0)| \geq \rho_0^{-2}$$

at  $(x_0, t_0)$ , and

$$r_0|v(x_0, t_0)| \geq \frac{1}{4} \sup_{x \in \mathbb{R}^3, t \leq t_0} r|v(x, t)|.$$

**Remark 1.2.** According to [8], if a smooth solution blows up in finite time, then the scaling invariant quantity  $r|v(x, t)|$  must also blow up in finite time. So, the condition in (b) can always be satisfied.

**Remark 1.3.** The factor  $1/4$  in the statement of the theorem can be replaced by any fixed positive number smaller than 1.

An important open question is to generalize the current result in (a) to the case when  $|v(x_0, t_0)|$  is very large but still much smaller than maximum.

Another question is: what happens when  $r_0|v(x_0, t_0)|$  is not large, but  $|v(x_0, t_0)|$  is large at almost maximal point  $(x_0, t_0)$ ?

**Remark 1.4.** The result and parameters in the theorem depend only on the norms of the initial value in (1.6). They do not depend on individual solutions.

## 2 Proof of Theorem 1.1

Let us prove part (a) first, after which the proof of (b) follows easily.

*Proof.* From the condition

$$\|v_0\|_{L^\infty(\mathbb{R}^3)} \leq N_0, \quad \|v_0\|_{L^2(\mathbb{R}^3)} \leq N_0, \quad |rv_0| \leq N_0,$$

by standard theory (see Proposition 4.1 in [8] e.g.), there exists a time  $h_0$  such that

$$\|v(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} \leq 2N_0, \quad t \leq h_0. \quad (2.1)$$

The proof is divided into several steps, using the method of contradiction.

*step 1.* setting up a limit solution.

Suppose part (a) of the theorem is false. Then for some  $\epsilon > 0$  and  $\sigma_0 > 0$ , there exists a sequence of solutions  $v_k$  with initial condition satisfying (1.6), defined on the time interval  $[0, T_k)$  for some  $T_k > h_0$ , which satisfies the following conditions.

(i) there exist sequences of positive numbers  $\rho_k \rightarrow 0$ , points  $x_k \in \mathbb{R}^3$ , and times  $t_k \in [0, T_k)$  such that

$$r_k|v_k(x_k, t_k)| \geq \rho_k^{-2};$$

(ii). for each  $k$ , the solution  $v_k$  in the parabolic region

$$P(x_k, t_k, [cQ_k]^{-1}) \equiv \{(x, t) \in [0, T_k) \mid |x_k - x| < (cQ_k)^{-1}, t_k - (cQ_k)^{-2} \leq t \leq t_k\}$$

is not, after scaling by the factor  $Q_k$ ,  $\epsilon$  close, in  $C^{2,1,\alpha}$  norm, to a nonzero constant vector. Here  $c = \sigma_0\epsilon$  and also

$$Q_k = |v_k(x_k, t_k)| \geq \frac{1}{4} \sup_{t \in [0, t_k], x \in \mathbb{R}^3} |v_k(x, t)|.$$

Write  $\alpha_k = r_k Q_k = r_k |v_k(x_k, t_k)|$ . We consider  $v_k$  in the space time cube

$$P(x_k, t_k, r_k/\sqrt{\alpha_k}) \equiv B(x_k, r_k/\sqrt{\alpha_k}) \times [t_k - (r_k/\sqrt{\alpha_k})^2, t_k].$$

Note that

$$\begin{aligned}\beta_k &\equiv \frac{r_k}{\sqrt{\alpha_k}} = \frac{r_k}{\sqrt{r_k Q_k}} = o(r_k), \\ Q_k \beta_k &= \sqrt{r_k Q_k} \rightarrow \infty, \quad k \rightarrow \infty.\end{aligned}\tag{2.2}$$

Define the scaled function

$$\tilde{v}_k = Q_k^{-1} v_k(Q_k^{-1} \tilde{x} + x_k, Q_k^{-2} \tilde{t} + t_k)\tag{2.3}$$

Then  $\tilde{v}_k$  is solution of the Navier-Stokes equation in the slab  $\mathbb{R}^3 \times [-(Q_k \beta_k)^2, 0]$ . Moreover, by the assumption on  $Q_k$ , we know that  $|\tilde{v}_k| \leq 4$  whenever defined. Since  $\tilde{v}_k$  are bounded mild solutions, we know from Proposition 4.1 in [8] e.g. that the  $C^{2,1,\alpha}$  norm of  $v_k$  are uniformly bounded in  $\mathbb{R}^3 \times [-(Q_k \beta_k)^2 + 1, 0]$ . In addition, the pressure  $P_k$ , satisfying  $\Delta P_k = \operatorname{div}(v_k \nabla_k)$ , also has uniformly bounded  $C_{\text{local}}^{2,1,\alpha}$  norm, by virtue of standard Schauder theory. Actually all  $C^{p,p/2}$  norms are bounded for  $p \geq 1$ . But we do not need this fact here.

Let us restrict the solution  $\tilde{v}_k$  to the cube

$$P(0, 0, Q_k \beta_k) = \{(\tilde{x}, \tilde{t}) \mid |\tilde{x}| \leq Q_k \beta_k, -(Q_k \beta_k)^2 \leq \tilde{t} \leq 0\}.$$

By the uniform bounds on  $C_{\text{local}}^{2,1,\alpha}$  norm and the fact that  $Q_k \beta_k \rightarrow \infty$ , we know there exists a subsequence, still called  $\{\tilde{v}_k\}$  that converges to an ancient solution of Navier-Stokes equation in  $C_{\text{local}}^{2,1,\alpha}$  sense. Let us call this ancient solution  $\tilde{v}$ . In the next step, we will show that it is 2 spatial dimensional solution, with one of the dimension being the  $z$  axis.

*step 2.* proving  $\tilde{v}$  is a 2D solution.

Denote by  $v_k^\theta$  the angular component of  $v_k$ . For the given initial value, it is known that

$$|v_k^\theta(x, t)| \leq \frac{N_0}{r}.$$

For  $x \in B(x_k, \beta_k)$ , we have, by (2.2),

$$|v_k^\theta(x, t)| \leq \frac{2N_0}{r_k}$$

when  $k$  is sufficiently large. Therefore

$$Q_k^{-1} |v_k^\theta(x, t)| \leq \frac{2N_0}{Q_k r_k} \rightarrow 0, \quad k \rightarrow \infty.\tag{2.4}$$

In the standard basis for  $\mathbb{R}^3$ , let  $x_k = (x_{k,1}, x_{k,2}, x_{k,3})$  with the third component being the one for the  $z$  axis, and let  $\xi_k = (0, 0, x_{k,3})$ . Since the vectors  $(x_k - \xi_k)/|x_k - \xi_k|$  are unit ones, there exists a subsequence, still labeled by  $k$ , which converges to a unit vector  $\zeta = (\zeta_1, \zeta_2, 0)$ . We use

$$\zeta, \zeta' = (-\zeta_2, \zeta_1, 0), (0, 0, 1)$$

as the basis of a new coordinate. Since this basis is obtained by a rotation around  $z$  axis, we know  $v_k$  is invariant. From now on, when we mention the coordinates of a point, we mean to use the new basis with the same origin. We still use  $(\theta, r, z)$  to denote the variables for the cylindrical system corresponding to this new basis.

For  $x \in B(x_k, \beta_k)$ , we recall that  $\theta$  is the angle between  $x$  and  $\zeta$ . Then

$$\cos \theta = \frac{(x - \xi_k) \cdot \zeta}{|x - (0, 0, x_3)^T|} = \frac{(x_k - \xi_k) \cdot \zeta}{|x_k - \xi_k|} + \frac{O(\beta_k)}{r_k} \rightarrow 1, \quad k \rightarrow \infty. \quad (2.5)$$

For  $v_k = v_k(x, t)$  in  $B(x_k, \beta_k) \times [t_k - \beta_k^2, t_k]$ , we have defined

$$\tilde{v}_k = \tilde{v}_k(\tilde{x}, \tilde{t}) = Q_k^{-1} v_k(Q_k^{-1} \tilde{x} + x_k, Q_k^{-2} \tilde{t} + t_k)$$

where  $x = Q_k^{-1} \tilde{x} + x_k$  and  $t = Q_k^{-2} \tilde{t} + t_k$ . Then for  $x = (x^{(1)}, x^{(2)}, x^{(3)})$  and  $\tilde{x} = (\tilde{x}^{(1)}, \tilde{x}^{(2)}, \tilde{x}^{(3)})$ , we have

$$\begin{cases} \partial_r v_k(x, t) = \partial_{x^{(1)}} v_k(x, t) \cos \theta + \partial_{x^{(2)}} v_k(x, t) \sin \theta \\ \quad = Q_k^2 \partial_{\tilde{x}^{(1)}} \tilde{v}_k(\tilde{x}, \tilde{t}) \cos \theta + Q_k^2 \partial_{\tilde{x}^{(2)}} \tilde{v}_k(\tilde{x}, \tilde{t}) \sin \theta \\ \partial_r^2 v_k(x, t) = Q_k^3 \partial_{\tilde{x}^{(1)}}^2 \tilde{v}_k(\tilde{x}, \tilde{t}) \cos^2 \theta + 2Q_k^3 \partial_{\tilde{x}^{(1)} \tilde{x}^{(2)}}^2 \tilde{v}_k(\tilde{x}, \tilde{t}) \sin \theta \cos \theta \\ \quad + Q_k^3 \partial_{\tilde{x}^{(2)}}^2 \tilde{v}_k(\tilde{x}, \tilde{t}) \sin^2 \theta; \\ \partial_z^2 v_k(x, t) = Q_k^3 \partial_{\tilde{x}^{(3)}}^2 \tilde{v}_k(\tilde{x}, \tilde{t}); \\ \partial_t^2 v_k(x, t) = Q_k^3 \partial_{\tilde{t}}^2 \tilde{v}_k(\tilde{x}, \tilde{t}). \end{cases} \quad (2.6)$$

For the pressure  $p_k = p_k(x, t)$ , recall that

$$\tilde{p}_k = \tilde{p}_k(\tilde{x}, \tilde{t}) = Q_k^{-2} p_k(Q_k^{-1} \tilde{x} + x_k, Q_k^{-2} \tilde{t} + t_k).$$

Therefore

$$\partial_r p_k(x, t) = Q_k^3 \partial_{\tilde{x}^{(1)}} \tilde{p}_k(\tilde{x}, \tilde{t}) \cos \theta + Q_k^3 \partial_{\tilde{x}^{(2)}} \tilde{p}_k(\tilde{x}, \tilde{t}) \sin \theta \quad (2.7)$$

Writing  $v_k = v_k^r e_r + v_k^\theta e_\theta + v_k^z e_z$ , then

$$\begin{aligned} v_k^r \partial_r v_k^r + v_k^z \partial_z v_k^r &= Q_k^3 [v_k^r(\tilde{x}, \tilde{t}) \partial_{\tilde{x}^{(1)}} \tilde{v}_k^r(\tilde{x}, \tilde{t}) \cos \theta \\ &\quad + v_k^r(\tilde{x}, \tilde{t}) \partial_{\tilde{x}^{(2)}} \tilde{v}_k^r(\tilde{x}, \tilde{t}) \sin \theta + \tilde{v}_k^z \partial_{\tilde{x}^{(3)}} \tilde{v}_k^r(\tilde{x}, \tilde{t})]. \end{aligned} \quad (2.8)$$

We substitute the above identities into the equation for  $v_k^r$ :

$$[\partial_r^2 + (1/r) \partial_r - \frac{1}{r^2}] v_k^r - (b \cdot \nabla) v_k^r + \frac{(v_k^\theta)^2}{r} - \frac{\partial p_k}{\partial r} - \frac{\partial v_k^r}{\partial t} = 0,$$

we arrive at

$$\begin{aligned} (\partial_{\tilde{x}^{(1)}}^2 + \partial_{\tilde{x}^{(3)}}^2) \tilde{v}_k^r - (\tilde{v}_k^r \partial_{\tilde{x}^{(1)}} + \tilde{v}_k^z \partial_{\tilde{x}^{(3)}}) \tilde{v}_k^r - \partial_{\tilde{x}^{(1)}} \tilde{p}_k - \partial_{\tilde{t}} \tilde{v}_k^r + \frac{1}{Q_k r} (\partial_{\tilde{x}^{(1)}} \tilde{v}_k(\tilde{x}, \tilde{t}) \cos \theta \\ + \partial_{\tilde{x}^{(2)}} \tilde{v}_k(\tilde{x}, \tilde{t}) \sin \theta) - \frac{1}{(Q_k r)^2} \tilde{v}_k^r + \frac{(r v_k^\theta)^2}{(Q_k r)^3} + O(\theta) = 0. \end{aligned}$$

Here the term  $O(\theta)$  stands for all those terms which vanish when  $\theta \rightarrow 0$  as  $k \rightarrow \infty$ . In particular all terms involving the derivative with respect to  $\tilde{x}^{(2)}$  are included in  $O(\theta)$ .

Recall that  $Q_k r$  is comparable to  $Q_k r_k$  which goes to  $\infty$ . Letting  $k \rightarrow \infty$  and noting that  $v_k$  and derivatives are uniformly bounded, we know that  $\tilde{v}^1$ , the limit of  $\tilde{v}_k^r$  satisfies

$$(\partial_{\tilde{x}^{(1)}}^2 + \partial_{\tilde{x}^{(3)}}^2)\tilde{v}^{(1)} - (\tilde{v}^{(1)}\partial_{\tilde{x}^{(1)}} + \tilde{v}^{(3)}\partial_{\tilde{x}^{(3)}})\tilde{v}^{(1)} - \partial_{\tilde{x}^{(1)}}\tilde{p} - \partial_{\tilde{t}}v^{(1)} = 0.$$

Here  $\tilde{v}^{(3)}$  is the limit of  $v_k^z$ , for which we have, in a similar manner

$$(\partial_{\tilde{x}^{(1)}}^2 + \partial_{\tilde{x}^{(3)}}^2)\tilde{v}^{(3)} - (\tilde{v}^{(1)}\partial_{\tilde{x}^{(1)}} + \tilde{v}^{(3)}\partial_{\tilde{x}^{(3)}})\tilde{v}^{(3)} - \partial_{\tilde{x}^{(3)}}\tilde{p} - \partial_{\tilde{t}}v^{(3)} = 0.$$

Note that  $\tilde{v}_k$  and its derivatives are uniformly bounded in the region of concern. When  $k \rightarrow \infty$ ,  $\theta \rightarrow 0$  in the region of concern. Hence  $\tilde{v}_k^\theta$  and derivatives all vanish when  $k \rightarrow \infty$ .

Finally we need to show that  $\tilde{v}^{(1)}$  and  $\tilde{v}^{(3)}$  are independent of the variable  $\tilde{x}^{(2)}$ . To prove it, let us recall that  $\partial_\theta v_k^r = \partial_\theta v_k^z = 0$ . Hence

$$-\partial_{x^{(1)}}v_k^r \sin \theta + \partial_{x^{(2)}}v_k^r \cos \theta = -\partial_{x^{(1)}}v_k^z \sin \theta + \partial_{x^{(2)}}v_k^z \cos \theta = 0.$$

This implies

$$\partial_{\tilde{x}^{(2)}}\tilde{v}_k = \partial_{\tilde{x}^{(1)}}\tilde{v}_k \tan \theta.$$

Taking  $k \rightarrow \infty$  ( $\theta \rightarrow 0$ ) we see the desired result.

*Step 3.*

Here we just use the fact that 2 dimensional ancient (mild) solutions are constants ([8]) and the regularity result Proposition 4.1 in the same paper to conclude that  $\tilde{v}_k$ , with  $k$  large, is  $\epsilon$  close to a nonzero constant vector in  $C_{\text{local}}^{2,1,\alpha}$  sense. This contradiction with the condition (ii) at the beginning of the section proves part (a) of the theorem.

Now we prove part (b).

Suppose part (b) of the theorem is false. Then for some  $\epsilon > 0$ , there exists a sequence of solutions  $v_k$  with normalized initial condition as above, defined on the time interval  $[0, T_k)$  for some  $T_k \in [h_0, T_0]$ , which satisfies the following conditions.

(i) there exist sequences of positive numbers  $\rho_k \rightarrow 0$ , points  $x_k \in \mathbb{R}^3$ , and times  $t_k \in [0, T_k)$  such that

$$r_k |v_k(x_k, t_k)| \geq \rho_k^{-2};$$

(ii). for each  $k$ , the solution  $v_k$  in the parabolic region

$$P(x_k, t_k, [cQ_k]^{-1}) \equiv \{(x, t) \in [0, T_k) \mid d(x_k, x, t_k) < (cQ_k)^{-1}, t_k - (cQ_k)^{-2} \leq t \leq t_k\}$$

is not, after scaling by the factor  $Q_k$ ,  $\epsilon$  close, in  $C_{\text{local}}^{2,1,\alpha}$  norm, to a nonzero constant vector. Here  $c = \sigma_0 \epsilon$  and also

$$r_k |v_k(x_k, t_k)| \geq \frac{1}{4} \sup_{t \in [0, t_k], x \in R^3} r |v_k(x, t)|.$$

Define as before  $Q_k = |v(x_k, t_k)|$ . Suppose  $k$  is large. Then for  $x \in B(x_k, \beta_k)$  with  $\beta_k = r_k/\sqrt{r_k Q_k} = o(r_k)$ , there holds, for  $t \leq t_k$ ,

$$r|v(x, t)| \leq r_k|v(x_k, t_k)| = r_k Q_k$$

and

$$r_k/2 \leq r \leq 2r_k$$

when  $k$  large. This shows, in the ball  $B(x_k, \beta_k)$  and for  $t \leq t_k$ ,

$$|v(x, t)| \leq 2Q_k.$$

Now we can scale by  $Q_k^{-1}$  in the above ball again as in the proof of part (a). By Theorem 2.8 of [11], the limit of scaled solutions is again a bounded, mild, ancient solution. Similar arguments as in part (a) lead to a contradiction, proving part (b).  $\square$

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